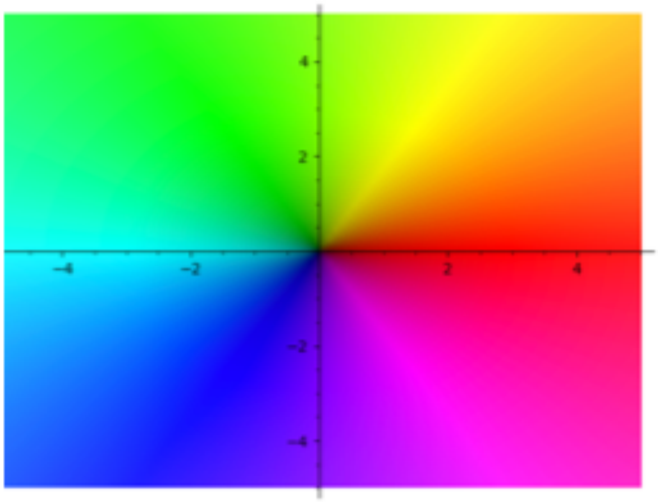
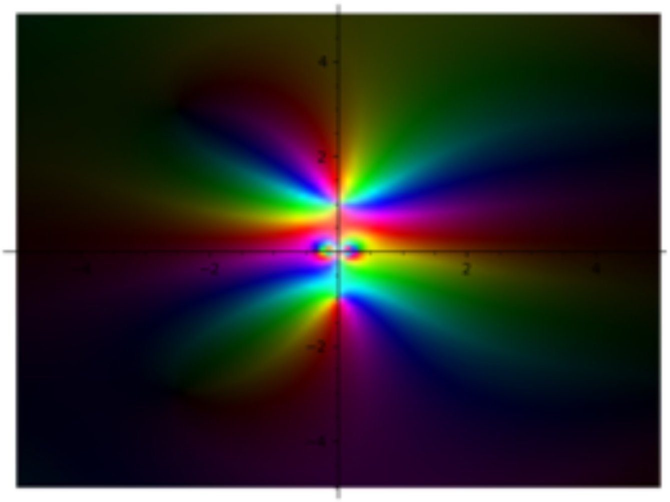
```
p1 = complex_plot(z,(-5,5),(-5,5))
p2 = complex_plot(f(z),(-5,5),(-5,5))
show(p1)
show(p2)
```

This fcn has singularities at $z = 0, i, -i, \frac{\pi}{2}$.

What type?

valuate





Answers: $z = \frac{\pi}{2}$ removable singularity —

$f(z) \to$ a specific #.

$z = i$ double pole $\frac{1}{(z-i)^2}$

$z = i + re^{i\theta}$ for small r

$f(z) \sim \frac{1}{r^2} e^{-2i\theta}$

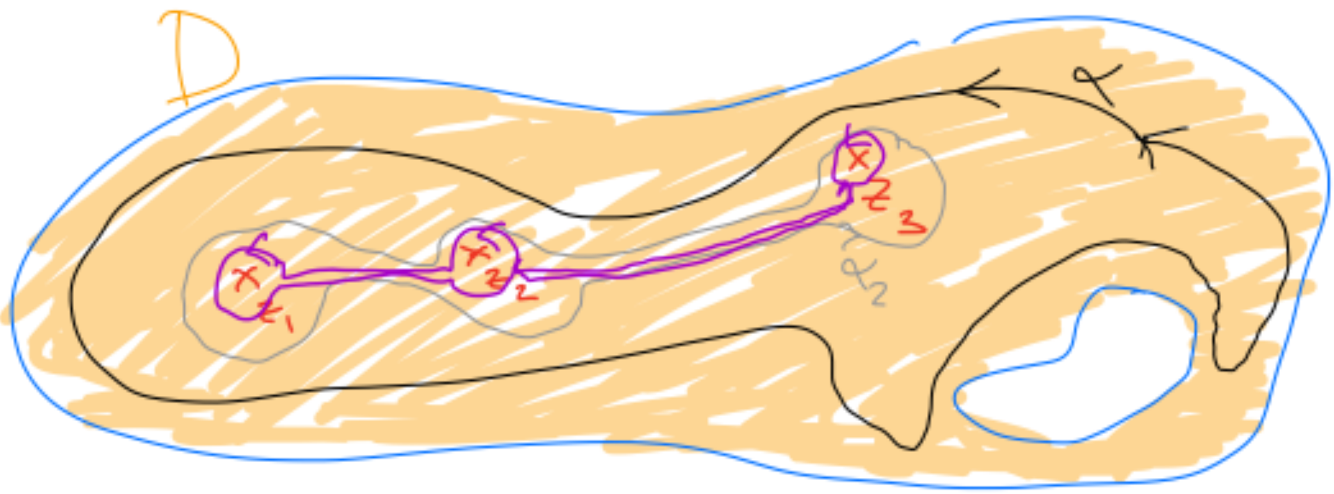
$z = -i$ single pole

$z = 0$ essential singularity!

Recall

$$\int_\alpha \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 2\pi i & \text{if } n=1 \\ 0 & \text{otherwise.} \end{cases}$$

Given any holomorphic fcn f on D with isolated singularities $z_1, z_2, \dots, z_n$; and suppose α is a simple, closed curve in D whose interior is in D s.t. $z_1, z_2, \dots, z_n$ are in the interior of α, then we can compute $\int_\alpha f(z)\,dz$.



$$\Rightarrow \int_{\substack{\alpha \\ \text{ccw}}} f(z)\,dz = \int_{\beta_1} f(z)\,dz + \dots + \int_{\beta_n} f(z)\,dz,$$

where each β_j is a ccw-oriented small circle around z_j.

For each j, we can expand $f(z)$ in its Laurent series around an annulus of the form $0 < |z - z_j| < \varepsilon_j$.

Near z_j, $f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_j)^k$

$$\Rightarrow \int_{\beta_j} f(z)\,dz = \int_{\beta_j} \sum_{k\in\mathbb{Z}} a_k (z-z_j)^k\,dz$$

absolutely &

Converges, unif on β_j

$$= \sum_{k\in\mathbb{Z}} a_k \underbrace{\int_{\beta_j} (z-z_j)^k\,dz}$$

$$= \begin{cases} 2\pi i & \text{if } k=-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= a_{-1}(2\pi i)$$

$\mathrm{Res}(f, z_j) = a_{-1}$

$$\therefore \int_{\alpha} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$

<u>Cauchy Residue Theorem!</u>

Given f holom. with isolated singularity z_0,

$\mathrm{Res}(f, z_0) := a_{-1}$, where

$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$ is the Laurent series corresponding to the annulus $0 < |z - z_0| < \varepsilon$ small pos. #.

(Eg if z_0 is a removable singularity, then $\text{Res}(f, z_0) = 0$.)

Examples: Calculating Residues

① $\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots -$

$\Rightarrow \text{Res}\left(\sin\left(\frac{1}{z}\right), 0\right) = 1.$

② $\dfrac{z+4}{(z+2)^3 + (z+2)^2} = g(z)$

Singularities: $(z+2)^3 + (z+2)^2 = 0$

$\Rightarrow (z+2)^2 (z+2+1) = 0$

$(z+2)^2 (z+3) = 0$

$$g(z) = \frac{z+4}{(z+2)^2(z+3)}$$

$$\operatorname{Res}\left(g(z), -3\right) = 1 .$$

How?: $$(z+3)\cdot\underbrace{\left(\frac{z+4}{(z+2)^2}\right)}_{\text{holom — just need const. term in Taylor series at } z=-3.} = \frac{1}{z+3}\left(\overbrace{\left(\frac{-3+4}{(-3+2)^2}\right)}^{1} + \xi(z+3) + \eta(z+3)^2 + \dots\right)$$

$$\operatorname{Res}\left(g(z), -2\right) = ?$$

$$\frac{1}{(z+2)^2}\left(\frac{z+4}{z+3}\right)$$

$z \to (z+2) - 2$

$$= \frac{1}{(z+2)^2}\left(\frac{(z+2)+2}{(z+2)+1}\right)$$

$$= \frac{1}{(z+2)^2}\left(\frac{z+2}{1+(z+2)} + \frac{2}{(z+2)+1}\right)$$

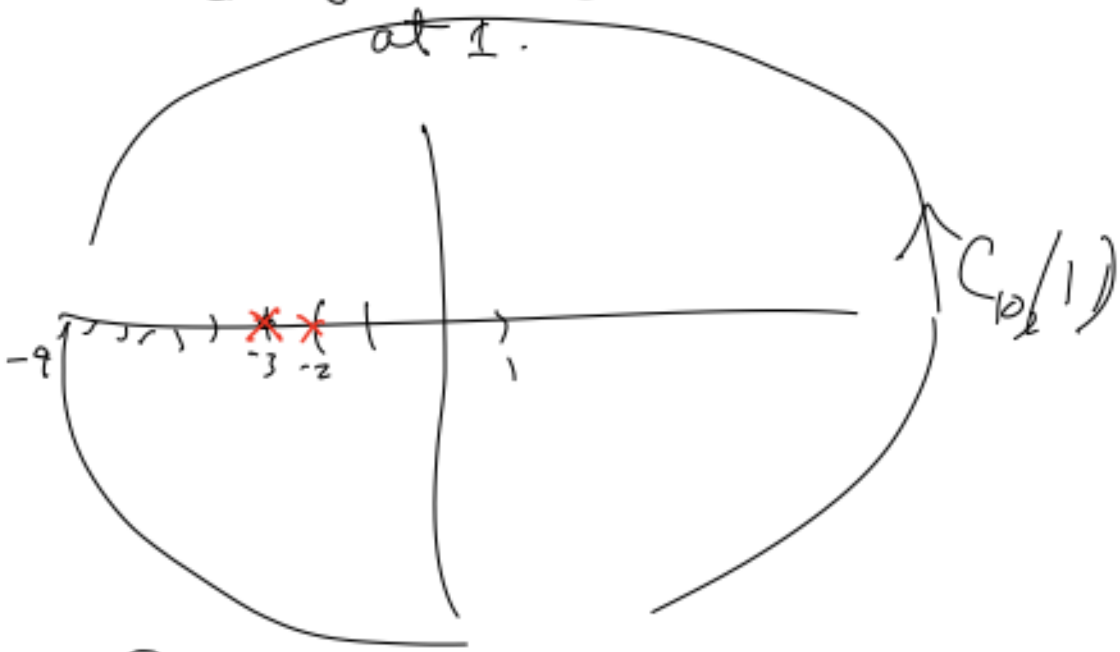
$$= \frac{1}{(z+2)^2}\left[(z+2)\left(\underline{1}+(-(z+2))+\dots\right) + 2\left\{1+(\underline{-(z+2)})+(-(z+2))^2+\dots\right\}\right]$$

$$\Rightarrow \mathrm{Res}(g,-2) = 1 + -2 = -1.$$

Eg of integral.

$$\int_{C_{10}(1)} \frac{z+4}{(z+2)^3+(z+2)^2}\,dz = 2\pi i\left[\overbrace{(-1)}^{\mathrm{Res}(g,-2)} + \overbrace{(1)}^{\mathrm{Res}(g,-3)}\right] = \boxed{0}.$$

$C_{10}(1)$ ← circle of radius 10 centered at 1.



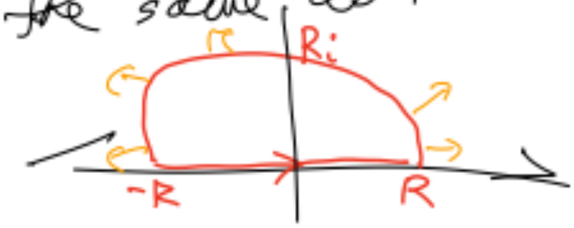
$$\int_{C_{10}(1)} \sin\left(\frac{1}{z}\right) dz = 2\pi i\,\underbrace{(1)}_{\mathrm{Res}\left(\sin\left(\frac{1}{z}\right),\,0\right)} = \boxed{2\pi i}.$$

$$\int_{C_{2.5}(0)} \frac{z+4}{(z+2)^3+(z+2)^2}\,dz = 2\pi i \,\mathrm{Res}(g,-2)$$
$$= 2\pi i(-1) = \boxed{-2\pi i}.$$

$$\int_{C_1(0)} \frac{z+4}{(z+2)^3+(z+2)^2}\,dz = 0.$$

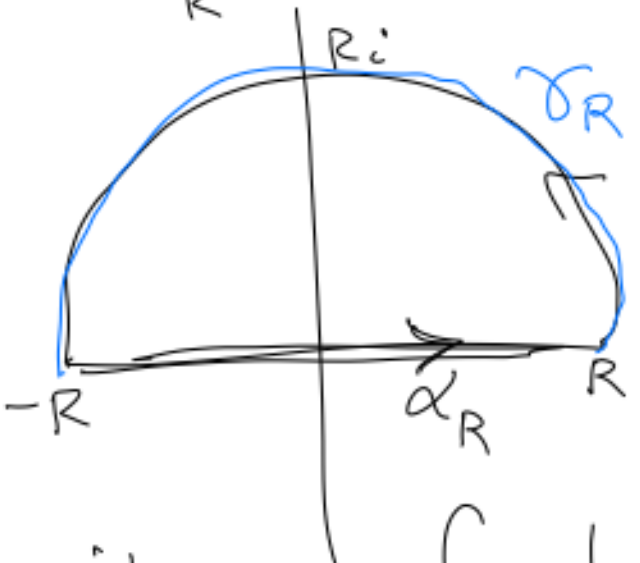
We can actually use the Residue theorem to calculate hard integrals on the real axis.

Idea: Make a contour that includes the real axis (or goes to it). Take a limit as the contour gets larger & larger — If the part that is not on the real axis goes to zero, then the limit is the same as the real integral.



$\boxed{\text{Example}}$ Find $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

Let C_R be this contour.



Let's integrate $\displaystyle\int_{C_R} \frac{1}{z^2+1}\,dz$.

$$= \int_{\alpha_R} \frac{1}{z^2+1}\,dz + \int_{\gamma_R} \frac{1}{z^2+1}\,dz$$

$$\int_{\alpha_R} \frac{1}{z^2+1}\,dz = \int_{-R}^{R} \frac{1}{x^2+1}\,dx$$

Notice for $R>1$

$$\left|\int_{\gamma_R} \frac{1}{z^2+1}\,dz\right| \le \int_{\gamma_R} \left|\frac{1}{z^2+1}\right| |dz|$$

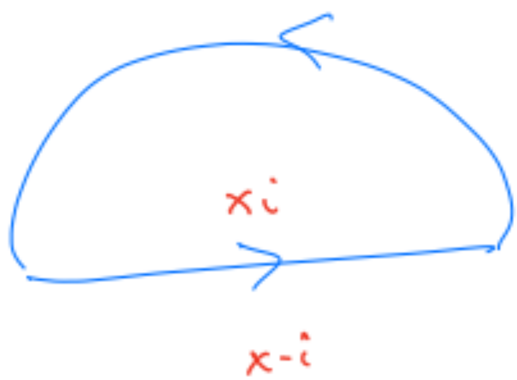
$$= \pi R \cdot \left(\frac{1}{R^2-1}\right) \to 0 \text{ as } R\to\infty.$$

$$\left|\frac{1}{z^2+1}\right| \le \frac{1}{|z|^2-1} = \frac{1}{R^2-1}$$

$$\int_{\gamma_R} \to 0 \text{ as } R \to \infty$$

$$\int_{\alpha_R} \to \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx \text{ as } R \to \infty.$$

$$\therefore \lim_{R\to\infty} \int_{C_R} \frac{1}{z^2+1}\,dz = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx.$$



$$\int_{C_R} \frac{1}{z^2+1}\,dz = 2\pi i \sum \text{Residues}$$

$$= 2\pi i \left(\mathrm{Res}\left(\frac{1}{z^2+1}, i\right)\right)$$

$$\mathrm{Res}\left(\frac{1}{z^2+1}, i\right):$$

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{z-i}\Big(\underbrace{\frac{1}{z+i}}_{\text{holo}}\Big)$$

$$\underset{\uparrow}{\_\_} + \_\_(z-i) + \_\_(z-i)^2$$

$$= \frac{1}{z-i}\left(\frac{1}{i+i} + \_\_(z-i) + \_\_(z-i)^2 + \dots\right)$$

$$\mathrm{Res} = \frac{1}{i+i} = \frac{1}{2i}$$

$$\therefore \int_{C_R} \frac{1}{z^2+1}\,dz = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

if $R > 1$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty}\int_{C_R} \frac{1}{z^2+1}\,dz$$

$$= \boxed{\pi}.$$

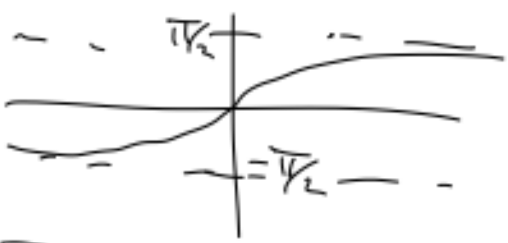
Check:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{1}{x^2+1}\,dx$$

$$= \lim_{R\to\infty}\left(\arctan(x)\Big|_{-R}^{R}\right)$$

$$= \lim_{R\to\infty} \arctan(R) - \arctan(-R)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}.$$



Other examples

$$\int_{x=-\infty}^{\infty} \frac{3x^3 - 2}{(x^2+1)(x^2+4)^2}\,dx = \cdots$$

$$= 2\pi i\left[\operatorname{Res}(i) + \operatorname{Res}(2i)\right]$$